Minimum-time running: a numerical approach

R. Maroński¹, K. Rogowski²

¹ Institute of Aeronautics and Applied Mechanics, ul. Nowowiejska 24, 00-665 Warszawa
² Ph.D. Studies, Faculty of Power and Aeronautical Engineering, Warsaw University of Technology, ul. Nowowiejska 24, 00-665 Warszawa

1. Introduction

The basic question referring to competitive running is: how should a runner vary his speed with the distance to minimize the time during which he covers a given distance? This problem may be considered using formalism of optimal control. Keller has solved the similar problem (maximization of the distance of the run for the given time) equating to zero the first variation of the functional [4, 5]. The same problem has been considered by Behncke for running and swimming [1], and by Cooper for wheelchair athletics [2]. More general models have been applied in theirs papers. The considerations are based on Pontryagin’s maximum principle. The proofs presented in both papers are very complicated. Maroński has solved the problem using the method of Miele (extremization of linear integrals using Green’s theorem) [6]. This method is relatively simple and it has a clear graphical interpretation therefore it has been recalled by Tözeren [9]. There is a weakness however; it may be applied only for relatively simple models of runner's motion. All methods mentioned above have a point in common – they base on analytical considerations. Restrictions typical for analytical methods do not appear in numerical approach. That is why the problem of minimum-time running is reconsidered in this presentation.

2. Method

The racer is regarded as a particle with the mass m. The vertical displacements of this particle associated with the cyclic nature of the stride pattern, and at the start of the race are neglected. The mathematical model follows from Newton’s second law [7]:

\[
\frac{dv}{dx} = \frac{f_{\text{max}}(v)}{v} \eta - \frac{r(v)}{v},
\]

and the equation of power balance

\[
\frac{dE}{dx} = \sigma(v) - f_{\text{max}}(v) \eta.
\]

where: v is runner’s velocity, x – covered distance (independent variable), f_{\text{max}}(v) – maximal propulsive force per unit mass, \( \eta \) - variable with the distance propulsive force setting, r(v) – resistance per unit mass, E – actual reserves of chemical energy per unit mass in excess of the non-running metabolism, \( \sigma(v) \) – recovery rate of chemical energy per unit mass, e(v) – the efficiency of transforming the chemical energy into mechanical one.

The propulsive force setting \( \eta \) (the control variable) is not known “a priori” and it is within the given interval

\[
0 \leq \eta \leq 1.
\]

The upper limit refers to maximal propulsive force that cannot be surpassed. The amount of energy E cannot be lower than zero during the race

\[
E(x) \geq 0.
\]

Boundary conditions should supplement the state equations (1) and (2). This model is known as the Hill-Keller model of running [8].

The problem is formulated as follows: Find v(x), \( \eta(x) \) and E(x) satisfying (1)÷(4) so that the time T of covering the given distance is minimized.
\[ T = \int_{0}^{D} \frac{1}{v} \, dx \Rightarrow \text{MIN}. \] (5)

Symbol D in upper limit of the functional (5) denotes the distance to be covered. The direct pseudospectral Chebyshev’s method [3] is applied in the presented approach. It employs N-th degree Lagrange polynomial approximations for state (v, E) and control (ν) variables. The values of these variables at the Chebyshev-Gauss-Lobatto points are expansion coefficients. The approach converts optimal control problem to a nonlinear programming problem, where unknown parameters are state and control variables at these points. The method is implemented in Matlab using sequential quadratic programming algorithm.

3. Results

As a verification of the numerical method the example referring to the 400 m run is given. The data are taken from [6].

![Fig. 1. The optimum velocity v versus covered distance x.](image)

4. Discussion

The proposed method confirms the earlier results of Keller [4, 5] and Maroński [6]. For races at distances longer than some critical value, the optimal strategy is to accelerate initially applying maximum propulsive force until a speed is achieved that is maintained throughout the race. This velocity is approximately constant. After depleting all energy resources the velocity decreases at the finish. Formulation of the problem is very flexible in the presented approach. Any inequality constraints imposed on control and state variables may be considered. The number of state and control variables may be greater than in Miele’s method where the problem should be reduced to extremization of a line integral in the (E, v)-plane, and that is not always possible. The method has a weakness, however. It is moderately stable. The results are acceptable for relatively large number of node points.

References